SUBGROUPS - I

OUTLINES

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SUMMARY

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INTRODUCTION

In Group Theory, a branch of Abstract Algebra, a Subgroup is a structure-preserving function in Groups.

COMPLEX:

Let G be a group and $H \neq \phi \subseteq G$ then H is called the Complex of G.

Any non-empty subset of a group G is called a complex.

Example:

1. H = {1} is a complex of a multiplicate group G = {1,-1}

2. The set of even integers is a complex of (Z,+).

MULTIPLICATION OF COMPLEXES

Let *H* and *K* are two complexes of a group *G* then the product *HK* is defined as

$$\mathbf{HK} = \{\mathbf{hk} / \mathbf{h} \in \mathbf{H}, \mathbf{k} \in \mathbf{K}\}$$

Note: 1. HK is also a complex of G.

2. Multiplication of Complexes is Associative.

i.e., $H(KL) = (HK)L$

INVERSE OF A COMPLEX

Let H be a complex of G, then the inverse of H is defined as $\mathbf{H}^{-1} = \{\mathbf{h}^{-1} / \mathbf{h} \in \mathbf{H}\}$

i.e., $\mathbf{h} \in \mathbf{H} \Rightarrow \mathbf{h}^{-1} \in \mathbf{H}^{-1}$

NOTE : **We know that** $(\mathbf{HK})^{-1} = \mathbf{K}^{-1}\mathbf{H}^{-1}$ **and** $(\mathbf{H}^{-1})^{-1} = \mathbf{H}$

SUBGROUP

Let $(G, \cdot)$ be a group and $H \subseteq G$. H is said to be a Subgroup of G if H is itself a group w.r.t the binary operation '$\cdot$' of G.

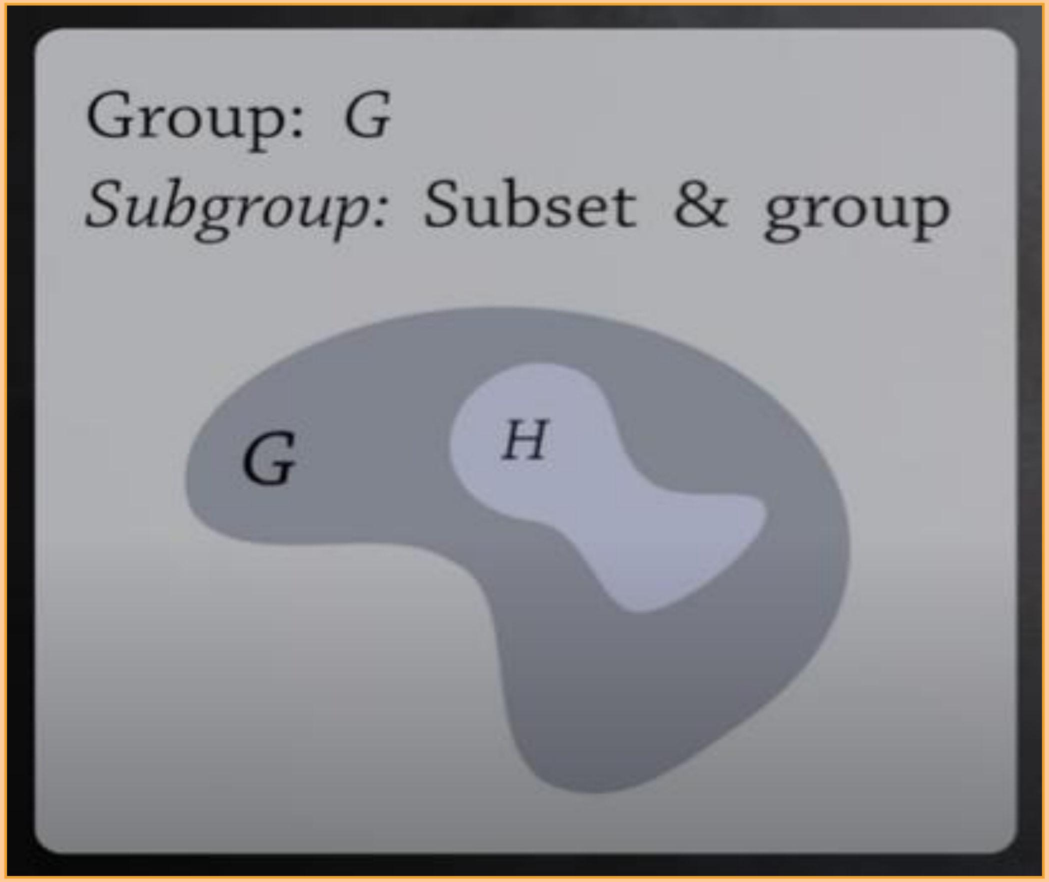
<u>EXAMPLES</u>

1. $(Z,+)$ and $(C,+)$ are two groups w.r.t addition and $Z \subseteq C \Rightarrow (Z,+)$ is a subgroup of $(C,+)$.

2. $(Q,+)$ and $(R,+)$ are two groups w.r.t addition and $Q \subseteq R \Rightarrow (Q,+)$ is a subgroup of $(R,+)$.

3. The set of Even integers is a subgroup of $(Z,+)$.

Relation between Group and Subgroup



Let G be group and H is a subgroup of G

$\Rightarrow$ 1. H is a subset of G. 2. H is a group.

IMPROPER (OR) TRIVIAL SUBGROUPS

Let G be a group and $e \in G$. Then the sets $\{e\}$ and G are Subgroups of G. These two Subgroups are called Improper Subgroups of G.

All other subgroups other than $\{e\}$ and G are called PROPER or Non-Trivial Subgroups of G.

EXAMPLE

We know that $G = \{1, -1, i, -i\}$ is a multiplicative Group.

$H_1 = \{1\}, H_2 = \{1, -1\}$ and $H_3 = G = \{1, -1, i, -i\}$ are three Subgroups of G. ($\because$ Identity element $e = 1$)

Clearly H_1 and H_3 are improper subgroups of G and H_2 is a proper subgroup of G.

Example for a Subgroup

Let $m \in Z$ and $mZ=\{ma / a \in Z\}$ then the set mZ is a Subgroup of a group $(Z,+)$.

Also mZ is cyclic Subgroup of $(Z,+)$ generated by m.

$$\therefore\ mZ=\{ma / a \in Z\} = \langle m \rangle$$

Example for a Subgroup

Let X be a non empty subset and $S(X)$ be the set of all bijective mappings of X onto itself under the composition of mappings. $\because S(X) = \{f /\ f \text{ is a bijection on } X\}$

For any $x_0 \in X$, we define $Hx_0 = \{f \in S(X) / f(x_0) = x_0\}$

then Hx_0 is a subgroup of $(S(X),0)$.

Q. What is the relation between between Complex and Subgroup ?

A. Eevery Subgroup is a Complex, but every Complex need not be a Subgroup.

Theorem : If H is a subgroup of a group G then $HH = H$

Proof : Let $x \in HH \quad \Rightarrow x = h_1h_2$, where $h_1 \in H, h_2 \in H$

$\therefore h_1 \in H, h_2 \in H \Rightarrow h_1h_2 \in H \qquad \Rightarrow x \in H$

$\therefore x \in HH \ \Rightarrow x \in H \qquad \Rightarrow HH \subset H \qquad \rightarrow (1)$

Again $h \in H, e \in H \quad \Rightarrow he \in HH \ \Rightarrow h \in HH$

$\therefore h \in H \ \Rightarrow h \in HH \qquad \Rightarrow H \subset HH \qquad \rightarrow (2)$

From (1) & (2), we get $HH \subseteq H$ & $H \subset HH \Rightarrow HH = H$

Hence proved

If H is a subgroup of a group G then $H^{-1} = H$

$$\mathbf{Proof:}\ \text{Let } G \text{ be a group and } H \subseteq G.$$

$$\mathbf{Part-I:}\ \text{Suppose } H \text{ is a subgroup of } G$$

$$\text{Let } x \in H^{-1} \Rightarrow x = h^{-1},\ \text{where } h \in H$$

$$\because h \in H \quad \Rightarrow h^{-1} \in H\ (\text{Inverse}) \Rightarrow x \in H$$

$$\therefore x \in H^{-1} \Rightarrow x \in H \quad \Rightarrow H^{-1} \subseteq H \rightarrow (1)$$

$$\mathbf{Part-II:}\ \text{Suppose } H^{-1} \subseteq H$$

$$\text{Let } h \in H \quad \Rightarrow h^{-1} \in H \quad \Rightarrow \left(h^{-1}\right)^{-1} \in H^{-1}$$

$$\Rightarrow h \in H^{-1}$$

$$\therefore h \in H \Rightarrow h \in H^{-1} \quad \Rightarrow H^{-1} \subseteq H \rightarrow (2)$$

$$\text{From } (1) \,\&\, (2) \text{ we get } H^{-1} = H$$

Hence proved

Note : The converse of the above theorem need not be true.

i.e., If $H = H^{-1}$ then H need not be a Subgroup of a group G.

Consider a set $H = \{1\}$ and a multiplicate group $G = \{1, -1\}$

Clearly $H = H^{-1} = \{1\}$ and H is a Subgroup of G . $\because (e = 1)$

Again Consider a set $H = \{-1\}$ and a multiplicate group $G = \{1, -1\}$

Clearly $H = H^{-1} = \{-1\}$, but H is a not a Subgroup of G as

H has not a multiplicative identity $e = 1$. $\because H^{-1} = \{(-1)^{-1} = -1\}$

Let H be a Subgroup of a group G. **<u>The identity element of H is same as the identity element of G.</u>**

A nonempty subset H of a group G is a subgroup of G iff $a, b \in H \Rightarrow ab^{-1} \in H$

Proof: Suppose H be a Subgroup of a group G and e is the identity element of G.

Let $a, b \in H \Rightarrow a^{-1}, b^{-1} \in H$ $\because$ inverse axiom

For $a \in H, b^{-1} \in H \Rightarrow ab^{-1} \in H$ $\because$ identity axiom

$\therefore a, b \in H \Rightarrow ab^{-1} \in H$

Conversely suppose that $a, b \in H \Rightarrow ab^{-1} \in H \rightarrow (1)$

1. <u>Identity axiom:</u> Let $a \in H \Rightarrow a \in G \left[\because aa^{-1} = a^{-1}a = e\right]$

For $a, a \in H \Rightarrow aa^{-1} \in H \Rightarrow e \in H$ $\left[\text{using } (1)\right]$

where e is the identity element in H.

2. **Inverse axiom:** Let $a \in H, e \in h\left[\because ae = ea = a\right]$

For $e, a \in H \Rightarrow ea^{-1} \in H \Rightarrow a^{-1} \in H \quad \left[\text{using } (1)\right]$

$\therefore a \in H \quad \Rightarrow a^{-1} \in H$

3. Associative axiom: Let $a, b, c \in H \Rightarrow a, b, c \in G$

$\Rightarrow a(bc) = (ab)c \quad \because H \subseteq G$ & G is a group.

$\therefore$ '.' is associative in H.

4. Closure axiom: Let $a, b \in H \Rightarrow a^{-1}, b^{-1} \in H$

$\therefore a \in H, b^{-1} \in H \Rightarrow a\left(b^{-1}\right)^{-1} \in H \Rightarrow ab \in H$

$\therefore$ '.' is a b inary operation in H

$\therefore$ H is group and $H \subseteq G \Rightarrow$ H is a subgroup of G.

Hence proved

Necessary and Sufficient Condtion for a Subgroup

Statement: A nonempty complex H of a group G is a subgroup of G if and only if

$(i)\ a,b \in H \Rightarrow ab \in H, \quad (ii)\ a \in H \Rightarrow a^{-1} \in H$

Proof: Suppose H be a Subgroup of a group G

$\Rightarrow$ **H is itself a group w.r.t the operation of G**

Let $a, b \in H \Rightarrow ab \in H$ $\quad \because$ **Closure axiom**

For $a \in H \Rightarrow a^{-1} \in H$ $\quad \because$ **inverse axiom**

Conversely suppose that

$a, b \in H \Rightarrow ab \in H \rightarrow (1)$ **and** $a \in H \Rightarrow a^{-1} \in H \rightarrow (2)$

1. **<u>Closure axiom:</u>** **Let** $a,b \in H \Rightarrow ab \in H$ $[\text{using } (1)]$

2. **<u>Associative axiom</u>:** **Let** $a,b,c \in H \Rightarrow a,b,c \in G$

$\Rightarrow a(bc) = (ab)c \qquad \therefore$ **G is a group &** $H \subseteq G$

3. **Existence of inverse:** Let $a \in H \Rightarrow a^{-1} \in H \quad (\because \text{using } 2)$

$\therefore$ Every element in H having multiplicative inverse in H.

4. **Identity axiom:** Let $a, b \in H \Rightarrow a^{-1}, b^{-1} \in H \quad (\because \text{using } 2)$

For $a, a^{-1} \in H \Rightarrow aa^{-1} \in H \Rightarrow e \in H \quad (\because aa^{-1} = a^{-1}a = e)$

$\Rightarrow$ 'e' is a the identity element in H.

$\therefore (H, .)$ is a group and $H \subseteq G \Rightarrow$ H is a Subgroup of G.

Hence proved

NOTE :

1. Let H be a complex of an multiplicative group $(G, .)$.

Then H is a Subgroup of $(G, .)$ iff $a,b \in H \Rightarrow ab^{-1} \in H$.

2. Let H be a complex of an additive group $(G, +)$.

Then H is a Subgroup of $(G, +)$ iff $a,b \in H \Rightarrow a - b \in H$.

S.T mZ is a subgroup of Z, where m is fixed +ve integer.

Sol: Let Z be the set of all integers and m is fixed +ve integer.

Let $x, y \in mZ$, where $mZ = \{ma / a \in Z\}$

$\Rightarrow x = ma$ **and** $y = mb$ **for** $a, b \in Z$

Now $x - y = ma - mb = m(a - b) = ma^1 \in mZ \quad \because a^1 = (a - b) \in Z$

$\Rightarrow$ **mZ is a subgroup of** $(Z, +)$.

Note: In general 3Z, 4Z, 5Z.....etc are subgroups of $(Z, +)$.

Example Problem:

Prove that the set of multiples of 3 is a subgroup of the group of integers under addition.

Here, the set of multiples of 3 = 3Z = $\{3a / a \in Z\}$

Union of two subgroups need not be a Subgroup.

We know that 2Z and 3Z are two Subgroups of $(Z,+)$.

$\because 2Z = \{\cdots\cdots -4,-2,0,2,4,\cdots\cdots\}$ & $3Z = \{\cdots\cdots -6,-3,0,3,6,\cdots\cdots\}$

$\Rightarrow$ Let $H = 2Z \cup 3Z = \{\cdots\cdots -6,-4,-3,-2,0,2,3,4,6,\cdots\cdots\}$

Let $2,3 \in H \Rightarrow 2-3 = -1 \notin H$ $\quad \Rightarrow$ H is not a subgroup of $(Z,+)$.

Intersection of two subgroups is also a Subgroup.

<u>Give an example of a nonabelian group</u>

Example: The $G = \{r_0, r_1, r_2, f_1, f_2, f_3\}$ of all symmetrices of an equilateral triangle forms a <u>nonabelian group</u> w.r.t to the composition of mappings 'o' on G.

1. Can an abelian group have a nonabelian Subgroup?

Ans: NO $\Rightarrow$ Every subgroup of an abelian is abelian.

2. Can a nonabelian group have an abelian Subgroup?

Ans: YES

Example-1: The set $\{e\}$ is a subgroup of a nonabelian group G, where e is the identity element of G.

Example-2: Let G be a nonabelian group and $x \in G,\ x \neq e$. Then $\langle x \rangle$ is an abelian subgroup of G. For example, S_3 is a nonabelian group such that only the cyclic subgroups are abelian. where S_3 is the permutation group on 3 symbols.

3. Can a nonabelian group have a nonabelian Subgroup?

Ans: YES

Note: The dihedral groups are examples of non – abelian groups

Every subgroup of an abelian group is abelian

Let G be an abelian group and H is a subgroup of G.

We prove that H is an abelian group

Let $a, b \in H \quad \Rightarrow a, b \in G \quad (\because H \subseteq G)$

$\Rightarrow ab = ba \quad (\because G$ is abelian group$)$

$\therefore$ H is a subgroup of G.

Find all subgroups of the group $(G = \{0,1,2,3,4,5\}, \oplus_6)$

$H_1 = \{0\}, \quad H_2 = \{0,3\}, H_3 = \{0,2,4\}, H_4 = \{0,1,2,3,4,5\} = G$

Here H_1 and H_4 are improper subgroups and H_2 and H_3 are proper subgroups of G.

Intersection of two subgroups of a group is also a subgroup

Proof: Let H and K are two subgroups of a group G.

Claim: We prove that $H \cap K$ is a subgroup of G

Let $a,b \in H \cap K \Rightarrow a,b \in H$ and $a,b \in K$

For $a,b \in H \Rightarrow ab^{-1} \in H \quad \because H$ is a subgroup of G

For $a,b \in K \Rightarrow ab^{-1} \in K \quad \because K$ is a subgroup of G

Using above, $ab^{-1} \in H$ and $ab^{-1} \in K \Rightarrow ab^{-1} \in H \cap K$

$\therefore H \cap K$ is a subgroup of G

NOTE: 1. H is a subgroup of a group G iff $a, b \in H \Rightarrow ab^{-1} \in H$

2. The intersection of any family of a subgroups of a group G is also a subgroup of G.

Let H and K be two subgroups of a group G. Then prove that $H \cup K$ is a subgroup of G iff $H \subseteq K$ or $K \subseteq H$

Proof: Let H and K be two subgroups of a group G.

PART-I: Suppose that $H \cup K$ is a subgroup of G

We prove that $H \subseteq K$ or $K \subseteq H$

If possible suppose that $H \not\subseteq K$ and $K \not\subseteq H$

$\because H \not\subseteq K$, then there exists $a \in H$ and $a \notin K \rightarrow (1)$

$\because K \not\subseteq H$, then there exists $b \in K$ and $b \notin H \rightarrow (2)$

From (1) & (2), $a \in H \Rightarrow a \in H \cup K$, $b \in K \Rightarrow b \in H \cup K$

$\therefore a \in H \cup K,\ b \in H \cup K \Rightarrow ab \in H \cup K$ (Closure)

$\Rightarrow ab \in H$ or $ab \in K$ or $ab \in H \cap K$

Case -I : $\underline{\text{Suppose that } ab \in H}$

$\because a \in H \Rightarrow a^{-1} \in H \text{ and } ab \in H \Rightarrow a^{-1}(ab) \in H \Rightarrow b \in H$

It is contradiction $\left(\because \text{ from } (2),\ b \notin H\right)$

$\therefore$ **Our supposition** $ab \in H$ **is wrong. i.e.,** $ab \notin H \rightarrow (3)$

Case -II : $\underline{\text{Suppose that } ab \in K}$

$\because b \in K \Rightarrow b^{-1} \in K \text{ and } ab \in K \Rightarrow (ab)b^{-1} \in K \Rightarrow a \in K$

It is contradiction $\left(\because \text{ from } (1),\ a \notin K\right)$

$\therefore$ **Our supposition** $ab \in K$ **is wrong. i.e.,** $ab \notin K \rightarrow (4)$

From $(3)\,\&\,(4), ab \notin H, ab \notin K \Rightarrow ab \notin H \cup K$ **(Closure failed)**

Hence our supposition $H \not\subseteq K$ **and** $K \not\subseteq H$ **is false.**

$\therefore H \subseteq K$ **or** $K \subseteq H$.

$\underline{\text{PART} - \text{II}}$: Suppose that $H \subseteq K$ **or** $K \subseteq H$.

$\because H \subseteq K \quad \Rightarrow K = H \cup K \quad \Rightarrow H \cup K$ **is a subgroup of G.**

$\because K \subseteq H \quad \Rightarrow H = H \cup K \quad \Rightarrow H \cup K$ **is a subgroup of G.**

Hence proved

Let H and K be two subgroups of a group G.

Then prove that HK is a subgroup of G iff $HK = KH$

Proof: Let H and K be two subgroups of a group G.

$\because$ **H is a subgroup of G** $\Rightarrow H = H^{-1}$ **and** $HH^{-1} = H \rightarrow (1)$

$\because$ **K is a subgroup of G** $\Rightarrow K = K^{-1}$ **and** $KK^{-1} = K \rightarrow (2)$

<u>PART – I</u> : Suppose HK is a subgroup of G.

$\because$ **HK is a subgroup of G** $\Rightarrow (HK)^{-1} = HK$

$$\Rightarrow (K^{-1}H^{-1}) = HK$$

$$\Rightarrow \quad KH = HK$$

PART-II: Suppose HK=KH

Now $(HK)(HK)^{-1} = (HK)(K^{-1}H^{-1}) = H(KK^{-1})H^{-1} = (HK)H^{-1}$

$$= (KH)H^{-1} = K(HH^{-1}) = KH = HK$$

$\therefore$ **HK is a subgroup of G.**

If H is a subgroup of a group G then $HH^{-1} \subseteq H$

Proof : Let G be a group and $H \subseteq G$.

Suppose H is a subgroup of G

Claim : <u>We prove that $HH^{-1} \subseteq H$</u>

Let $x \in HH^{-1} \Rightarrow x = ab^{-1}$, where $a \in H$, $b^{-1} \in H^{-1}$

$\because b^{-1} \in H^{-1} \Rightarrow b \in H \quad \Rightarrow b^{-1} \in H \left(\because \text{ H is a subgroup}\right)$

$\therefore a \in H,\ b^{-1} \in H \quad \Rightarrow ab^{-1} \in H \quad \Rightarrow x \in H$

$\therefore x \in HH^{-1} \Rightarrow x \in H \quad \Rightarrow HH^{-1} \subseteq H$

Hence proved

ADDITIONAL INFORMATION

The following web links are very useful to develop the subject skills and learning the problem solving techniques in easy manner.

https://web.ma.utexas.edu/users/rodin/343K/Subgroups.pdf

http://www.maths.lth.se/matematiklth/personal/ssilvest/AlgebraVT2011/ Lecture09Silvestrov.pdf

https://www.youtube.com/watch?v=TJAQNIGvfjE

https://www.youtube.com/watch?v=dpj2dVOscI0

http://web.math.princeton.edu/swim/SWIM%202010/SWIM%202010%20Course%20II%20Lecture%20Notes%206%20of%207.pdf

https://www.youtube.com/watch?v=aWxznOOMPsk

SUMMARY

I have concluded that PPT presentation is very useful in establishing objectives, Basic Concepts and illustrating concrete examples.

I hope that utilizing all of these methods through PPT Slides helps to engage students with different types of learning styles.

I have added definitions, example problems and theorems of the chapter Subgroups in brief and short methods.

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